Sitting in the back of Ms. Corey’s sixth-grade mathematics class, I enjoyed seeing students enthusiastically demonstrate their understanding of absolute value. On the giant number line on the classroom floor, they counted the steps that they needed to take to get back to zero. The old definition of absolute value as a number as its distance from zero—learned by students and teachers of the previous generation—has long ago been replaced with this algebraic statement: $|x| = x$ if $x \geq 0$ or $-x$ if $x < 0$. The absolute value learning objective in high school mathematics requires students to solve far more complex absolute value equations and inequalities.

What has been the result of this more conceptual approach? We have witnessed an increased sense of competence and confidence in students. They can make sense of and correctly solve absolute value problems. In classroom after classroom where this approach has been introduced, students have demonstrated mastery and achieved competency more quickly than students taught through the traditional two-case approach. Collaborating with colleagues and getting student feedback has allowed for valuable refinements and resulted in the method described here.

Let’s first examine the approach that is commonly used to introduce students to solving absolute value equations and inequalities. Consider the following problem: $|2x - 1| = 7$. Typically, students are taught to make this into two equations, $2x - 1 = 7$ and $2x - 1 = -7$, and solve each for $x$. The justification given is that since absolute value means “distance from zero,” this distance could be in either a positive or a negative direction. Although mathematically valid, this method does not make sense conceptually beyond the simple case of $|x| = c$ where $c$ is any positive real number.

However, I cannot remember students attacking the task with enthusiasm or having any understanding beyond “make the inside positive.”

The desire to rekindle in students the conceptual understanding and mathematical success with absolute value seen in Ms. Corey’s classroom prompted consideration of a different approach to the topic: combining the number-line distance definition with the idea of using transformations of a “parent” function. This approach connects the visual representation of the number line and the verbal context of distance from a fixed point to the abstract symbolic form, thus preventing the process from becoming a series of steps to be memorized and followed blindly.

We suggest that the absolute value concept can be more powerfully leveraged by teaching students to use a transformational approach to make sense of the problem.
of the meaning of the solution set. This method has been used with middle school and high school students with remarkable success. When students understand the meaning of absolute value and can transform an equation or inequality so that the coefficient of the variable is 1, they are able to find the solution set easily by (a) determining the location of the critical points that are equidistant from the “anchor point” (which represents an offset from zero) and (b) identifying whether the solutions fall on, between, or beyond these points. What follows is a primer for using this approach.

INTRODUCING THE CONCEPT OF ABSOLUTE VALUE
As mentioned, students’ first introduction to absolute value is as the distance from zero. Thus, |−5| = 5 and |3| = 3 might be interpreted, respectively, as “the distance from −5 to zero is 5” and “the distance from 3 to zero is 3.” This interpretation provides a conceptual basis for understanding absolute value. However, many textbooks (and teachers) will move away from this conceptual basis when students begin learning to solve absolute value equations and inequalities. The transformational approach extends the initial conceptual understanding to realizing that the distance between two values is the absolute value of their difference—that is, the distance from \( x \) to \( b \) is \(|x - b|\).

WORKING WITH ABSOLUTE VALUE EQUATIONS
Students must first learn to connect the symbolic expression \(|x - b| = c\) to the verbal phrase “\( x \) is \( c \) units from \( b \) in either direction.” One analogy that teachers have used to help students with this concept is to talk about friends’ homes being a certain distance apart. For instance, “My friend Khiem lives at 15 Sycamore Lane, and I live 8 houses away [assuming that house numbers change by units of 1]. Where could my house be located?” Students readily figure out that the solution is a matter of moving 8 units from 15 in either the positive or the negative direction. To help students connect this idea to absolute value, ask them to make a visual representation of this problem on a number line (see fig. 1).

Next, ask students how we might write a mathematical equation for this visual representation. With some help, students are usually able to come up with \(|x - 15| = 8\) as a translation of the sentence “The distance from my house \( x \) to house 15 is 8.” In this way, students have an opportunity to connect a meaningful context to a visual model, a verbal description, and a symbolic representation, thus leading to deeper conceptual understanding (NCTM 2000). The appendix (p. 596) provides practice with this idea. (These additional problems are also posted online at www.nctm.org/mt.)

When creating the visual model on a number line, it is essential to locate the “anchor point,” \( b \). Conceptually, this anchor point represents the “offset” from zero and is the point from which the two critical values are found by moving \( c \) units in the positive and negative directions (see fig. 1).
Students can then write the solution from the critical values plotted: $x = \{b - c, b + c\}$. This conceptual approach first solves the equation visually as distances on a number line and then records the solution in algebraic notation if needed.

Following this introduction of the case $|x - b| = c$ when $b > 0$, students are asked to consider a problem in the format $|x + b| = c$. For instance, how might $|x + 5| = 2$ be interpreted verbally as a problem of distance? Given time to discuss their thinking in pairs or small groups, students will typically arrive at the idea that the expression $x + 5$ can be thought of as $x - (-5)$, thus establishing the anchor point of $-5$ from which some distance is being measured. If students do not readily arrive at this idea, it may be helpful to ask a simpler question: “When $x$ is located at the anchor point, the distance between $x$ and the anchor point is 0. Determining the anchor point is equivalent to determining the value of $x$ such that $|x + 5| = 0$. What value of $x$ will make the absolute value equation equal 0?” From here, students are quickly able to determine that the anchor point is $-5$.

Transforming the original equation, we get $|x - (-5)| = 2$, which can be expressed verbally as “$x$ is 2 units from $-5$ in either direction.” Visually, the anchor point is placed at $-5$, and the solutions are found by moving 2 units in either direction (see fig. 3). Algebraically, $x = \{(-5 - 2), (-5 + 2)\}$ or, simplified, $x = \{-7, -3\}$.

The connection between absolute value and distance also provides an unexpected benefit: It eliminates the difficulty students have in equating $|x - b|$ and $|b - x|$. Students whose understanding is restricted to the procedural algorithm must mentally think through the distributive property and the property of opposites and apply the definition of absolute value before they can grasp or retain the equivalence:

$$|x - b| = |x - b - b + b|$$
$$= |x - b + b - b|$$
$$= |b - x|$$

In contrast, when set in the context of distance, students naturally “know” that these expressions are equal, just as they “know” without question that the distance from home to school is the same as the distance from school back to home.

**WORKING WITH ABSOLUTE VALUE INEQUALITIES**

This work extends naturally to absolute value inequalities. Again, there is no need to create two inequalities when using the transformational approach. Take a simple example: $|x - 3| > 4$. This is read as, “The distance from $x$ to 3 is greater than 4.” Visually, it is modeled as shown in figure 4.

From the representation, students can determine that the solution is $x > 7$ or $x < -1$. When the inequality symbol is the is-less-than symbol, the problem becomes finding points whose distance is less than a given value from the anchor point. Students readily catch on to this approach as an extension to their understanding of absolute value equations. The appendix (p. 597) provides practice with this idea. Encourage students to check their solutions by substituting values within each of the graph’s intervals, a technique they will use later when they learn to examine critical points of a function in calculus.

**HANDLING NONZERO COEFFICIENTS OTHER THAN 1**

When the coefficient (let’s call this $A$) of the variable is other than 1, it is simplest for students to treat $A$ if it were 1 to begin with. This approach can also be thought of as substituting a single variable with coefficient 1 for something more complex, a strategy students will use later in trigonometry and calculus. The problem is then worked out in the manner described above. Then, because these solutions represent not $x$ but $Ax$, the original coefficient is brought back. The last step is to divide each solution by the coefficient, $A$.

Let’s examine this approach with the example $|3x - 7| = 5$. First, take $3x$ to be a single unit and
think, “The distance from 3x to 7 is 5.” This step leads to the representation shown in figure 5.

This represents the solution for 3x, so the solutions 2 and 12 are then divided by the coefficient 3 to obtain \( x = \{2/3, 4\} \) (with each of these representing a distance of 5/3 from 7/3). We have found that students have more success doing the division after setting up the visual representation with the coefficient intact. Errors from incorrectly adding or subtracting fractional values are virtually eliminated.

The variable substitution approach also eliminates errors that commonly result when the coefficient of \( x \) is negative. Students read problems in the form \(|b - Ax| > c\) as they would those in the form \(|Ax - b| > c\)—that is, as “Ax is farther than \( c \) units away from \( b \).” They also realize that because \( Ax + b = b + Ax \) by the commutative property of addition, \(|b + Ax| < c \) and \(|Ax + b| < c\) may both be understood as the values of \( Ax \) that are closer than \( c \) units from \( -b \). The appendix (p. 598) provides practice in solving absolute value inequalities when the coefficient of \( x \) is not equal to 1.

EXTENDING STUDENTS’ UNDERSTANDING

As a way to allow students to develop a sense of ownership of their newly formed knowledge, ask them to build their own absolute value equality or inequality and show the solution set. The following steps can be used for this process. Afterward, have students challenge one another to solve the equations or inequalities created by their classmates.

Step 1: Choose a “parent” inequality centered at zero.
Step 2: Use a coefficient to expand (stretch) or compress (shrink) the solution set.
Step 3: Use a constant to translate (slide) the solution set.
Step 4: Verify that the final solution set matches the original equality or inequality created.

After students explore the impact of coefficients between 0 and 1 and those greater than 1, ask them to generate a conjecture about what happens and a justification as to why. A further extension asks students to use the visual representation on the number line and the verbal descriptions that they generated to explain conceptually why the solution set for \(|b - Ax| > c\) is equal to that of \(|Ax - b| > c\).

CONNECTING THE TRANSFORMATIONAL APPROACH TO LATER TOPICS

The transformational approach, which builds understanding by starting from a parent function, is often advocated for students’ learning about functions (e.g., Heid and Blume 2008; Ward 2001). The approach we describe here precedes students’ formal study of functions and can potentially set the stage for later applications of such an approach. Teachers who have used this approach have reported tremendous success with students, who have commented that this approach is often perceived to be “too easy to be math.”

REFERENCES


For PDFs of the additional problems given in the appendix, go to the Mathematics Teacher Web site: www.nctm.org/mt.
Introduction to Absolute Value Equations

I know that …
- The absolute value of a number means its distance from zero.
- The distance from \( x \) to \( b \) is equivalent to the distance from \( b \) to \( x \) because \( |x - b| = |b - x| \).
- If \( |x - b| = c \), then \( x \) is \( c \) units from \( b \) in either direction.

For each equation, write a sentence in words, draw a graph, and identify the solution(s).

1. \( |x| = 3 \)
   - Words: The distance from \( x \) to 0 is ______.
   - \( x = \) _____ or \( x = \) _____

2. \( |x| = 2 \)
   - Words: __________________________
   - \( x = \) _____ or \( x = \) _____

3. \( |x - 0| = 5 \)
   - Words: __________________________
   - \( x = \) _____ or \( x = \) _____

4. \( |x - 4| = 2 \)
   - Words: __________________________
   - \( x = \) _____ or \( x = \) _____

5. \( |4 - x| = 2 \)
   - Words: __________________________
   - \( x = \) _____ or \( x = \) _____

6. \( |x + 1| = 5 \)
   - Words: __________________________
   - \( x = \) _____ or \( x = \) _____

7. \( |x + 2| = 3 \)
   - Words: __________________________
   - \( x = \) _____ or \( x = \) _____

8. \( |x - 5| = 0 \)
   - Words: __________________________
   - \( x = \) _____ or \( x = \) _____

9. \( |x + 3| = -4 \)
   - Words: __________________________
   - \( x = \) _____ or \( x = \) _____

10. \( |x - b| = c \)
    - Words: _______________________________________________________________________________________
    - Which value is plotted first? _____ Which value tells the distance? ____ Graph the solution.
    - \( x = \) _____ or \( x = \) _____
Absolute Value Inequalities

I know that …
• The distance from \(x\) to \(b\) is equivalent to the distance from \(b\) to \(x\) because \(|x - b| = |b - x|\).
• If \(|x - b| < c\), then \(x\) is less than \(c\) units from \(b\) in either direction.
• If \(|x - b| > c\), then \(x\) is greater than \(c\) units from \(b\) in either direction.

For each inequality, write a sentence in words, draw a graph, and identify the solution(s).
1. \(|x| > 2\)
2. \(|x - 0| < 5\)
3. \(|x - 3| > 4\)

Words: The distance from \(x\) to 0 is ...
Solution: \(x < \) or \(x > \)

4. \(|x + 1| < 3\)
5. \(|x - 4| \geq 2\)
6. \(|x + 1| \leq 4\)

Words: ...
Solution: ...

7. \(|x - 2| > 3\)
8. \(|2 - x| > 3\)
9. \(|x + 3| > -1\)

Words: ...
Solution: ...

10. Graph the solutions for \(x\) if \(|x - b| < c\).
Which value (\(b\) or \(c\)) is plotted first? ____
Which value tells the distance? ____
Explain why the possible \(x\)-values are in one continuous region or two separate regions. ______________

11. Graph the solutions for \(x\) if \(|x - b| \geq c\).
Explain why the possible \(x\)-values are in one continuous region or two separate regions. ______________
More Complicated Inequalities

Example: \[ |5 - 2x| \leq 3 \]

Graph \(2x\): All points 3 or fewer units away from 5; \(5 - 3 = 2\) and \(5 + 3 = 8\) mark the endpoints.
Solution for \(2x\): All points between and including 2 and 8.
Graph \(x\). Divide the solution for \(2x\) by 2: \(2/2 = 1\) and \(8/2 = 4\).
Solution for \(x\): All points between 1 and 4, inclusive.

\[ |5 - 2x| \leq 3 \rightarrow 5 - 3 \leq 2x \leq 5 + 3 \]
\[ 2 \leq 2x \leq 8 \]
\[ 1 \leq x \leq 4 \]

Solve the following absolute value inequalities first for \(Ax\) and then for \(x\). Draw the graph and write the algebraic solution statement for each of the following problems:

1. \(|3x| > 2\)
Solution: ________________

2. \(|2x - 0| < 5\)
Solution: ________________

3. \(|4x - 2| \geq 6\)
Solution: ________________

4. \(|2x - 1| > 3\)
Solution: ________________

5. \(|3x - 4| \leq 4\)
Solution: ________________

6. \(|4 - 3x| \leq 4\)
Solution: ________________

7. \(|6x + 2| > 4\)
Solution: ________________

8. \(|7 - 5x| < 3\)
Solution: ________________

9. \(|2x + 3| > 5\)
Solution: ________________
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APPENDIX
Introduction to Absolute Value Equations

I know that …

• The absolute value of a number means its distance from zero.
• The distance from $x$ to $b$ is equivalent to the distance from $b$ to $x$ because $|x - b| = |b - x|$.
• If $|x - b| = c$, then $x$ is $c$ units from $b$ in either direction.

For each equation, write a sentence in words, draw a graph, and identify the solution(s).

1. $|x| = 3$
   Words: The distance from $x$ to 0 is ________.
   $x = _____$ or $x = _____$

2. $|x| = 2$
   Words: _______________________
   $x = _____$ or $x = _____$

3. $|x - 0| = 5$
   Words: _______________________
   $x = _____$ or $x = _____$

4. $|x - 4| = 2$
   Words: _______________________
   $x = _____$ or $x = _____$

5. $|4 - x| = 2$
   Words: _______________________
   $x = _____$ or $x = _____$

6. $|x + 1| = 5$
   Words: _______________________
   $x = _____$ or $x = _____$

7. $|x + 2| = 3$
   Words: _______________________
   $x = _____$ or $x = _____$

8. $|x - 5| = 0$
   Words: _______________________
   $x = _____$ or $x = _____$

9. $|x + 3| = -4$
   Words: _______________________
   $x = _____$ or $x = _____$

10. $|x - b| = c$
    Words: _______________________
    Which value is plotted first? _____ Which value tells the distance? _____ Graph the solution.
    What can $x$ equal? $x = _____$ or $x = _____
**Absolute Value Inequalities**

I know that …
- The distance from $x$ to $b$ is equivalent to the distance from $b$ to $x$ because $|x - b| = |b - x|$.  
- If $|x - b| < c$, then $x$ is less than $c$ units from $b$ in either direction.  
- If $|x - b| > c$, then $x$ is greater than $c$ units from $b$ in either direction.

For each inequality, write a sentence in words, draw a graph, and identify the solution(s).

1. $|x| > 2$  
2. $|x - 0| < 5$  
3. $|x - 3| \geq 4$

- Words: The distance from $x$ to 0 is  
- Solution: $x < ____$ or $x > ____$

4. $|x + 1| < 3$  
5. $|x - 4| \geq 2$  
6. $|x + 1| \leq 4$

- Words:  
- Solution:  

7. $|x - 2| > 3$  
8. $|2 - x| > 3$  
9. $|x + 3| > -1$

- Words:  
- Solution:  

10. Graph the solutions for $x$ if $|x - b| < c$.  
Which value (b or c) is plotted first?  
Which value tells the distance?  
Explain why the possible $x$-values are in one continuous region or two separate regions.  

11. Graph the solutions for $x$ if $|x - b| \geq c$.  
Explain why the possible $x$-values are in one continuous region or two separate regions.
More Complicated Inequalities

Example: \(|5 - \frac{2}{x}| \leq 3\)

Graph \(\frac{2}{x}\): All points 3 or fewer units away from 5; \(5 - 3 = 2\) and \(5 + 3 = 8\) mark the endpoints

Solution for \(\frac{2}{x}\): All points between and including 2 and 8.

Graph \(x\): Divide the solution for \(\frac{2}{x}\) by 2: \(2/2 = 1\) and \(8/2 = 4\).

Solution for \(x\): All points between 1 and 4, inclusive

\(|5 - \frac{2}{x}| \leq 3 \rightarrow 5 - 3 \leq \frac{2}{x} \leq 5 + 3\)
\[\frac{2}{x} \leq 8\]
\[1 \leq x \leq 4\]

Solve the following absolute value inequalities first for \(Ax\) and then for \(x\). Draw the graph and write the algebraic solution statement for each of the following problems:

1. \(|3x| > 2\)

Solution: ________________

2. \(|2x - 0| < 5\)

Solution: ________________

3. \(|4x - 2| \geq 6\)

Solution: ________________

4. \(|2x - 1| > 3\)

Solution: ________________

5. \(|3x - 4| \leq 4\)

Solution: ________________

6. \(|4 - 3x| \leq 4\)

Solution: ________________

7. \(|6x + 2| > 4\)

Solution: ________________

8. \(|7 - 5x| < 3\)

Solution: ________________

9. \(|2x + 3| > 5\)

Solution: ________________